Int. J. Solids Structures Vol. 19, No. 11, pp. 1009–1017, 1983
Printed in Great Britain.

0020-7683/83 $3.00+.00
Pergamon Press Ltd.

A GROWTH LAW FOR PROPAGATION OF ARBITRARY SHAPED DELAMINATIONS IN LAYERED PLATES†

W. J. Bottega
Section of Applied Mechanics, Yale University, P.O. Box 2157, Yale Station, New Haven, CT 06520, U.S.A.

(*Received* 16 *November* 1982; *in revised form* 25 *March* 1983)

Abstract—A growth law is derived which governs the propagation of a delamination embedded in a layered plate, with the shapes of the delaminated area and of the plate assumed to be arbitrary. The plate is considered to be subjected to edge loading.

INTRODUCTION

Debonded regions between layers of structural elements made of composite materials are referred to as delaminations. In this paper we are concerned with problems involving growth of delaminations in edge loaded layered plates such as that of an elliptic delamination embedded in a compressively loaded layered rectangular plate. Although the problem is essentially a three dimensional one, the phenomenon of delamination growth can be modeled by nonlinear plate theory together with a growth law in the spirit of the DCB models of crack growth in mode *I* fracture. An example of such an approach can be found in Bottega and Maewal[1] wherein a one-dimensional model problem involving a penny shaped delamination in a circular plate was considered. It is to the derivation of the growth law for the associated two dimensional moving intermediate boundary problem that the present work is addressed.

Starting with the potential energy functional for the plate, the corresponding form of the governing partial differential equations and boundary conditions are derived by using the theorem of stationary potential energy coupled with a Griffith type fracture criterion where the mobility of the moving intermediate boundary, the delamination edge, is taken into account. The general form of the growth law is found as a consequence of the transversality condition associated with the moving intermediate boundary.

Explicit forms of the general growth law for the class of problems of interest are found by evaluating the strain energy density using a geometrically nonlinear plate theory to describe the deformation of the layers. These reveal some interesting characteristics concerned with the factors affecting growth.

GENERAL FORMS OF THE GROWTH LAW AND GOVERNING EQUATIONS

Consider a layered plate occupying the region R with outer boundary S_0 which contains a region of delamination R_{I_0} bounded by the curve S_{I_0} (see Fig. 1). When the plate is subjected to the generalized forces $\mathbf{P}(s)$ on S_0, resulting in the generalized displacement field $\hat{\mathbf{u}}(\mathbf{x})$, the region of delamination grows to R_I with boundary S_I where $R_{I_0} \in R_I \in R$ and x_i are Cartesian coordinates in the plane of the delamination. The bonded region of the plate is denoted by R_0 such that $R_I \cup R_0 = R$.

In what follows, delamination growth is assumed to be governed by a Griffith type criterion such that an energy release of Γ is required to produce a unit area of new delamination and Γ is a characteristic of the bonding agent and layering material.

Let us define the energy functional Π as‡

$$\Pi = \mathscr{G}\{\hat{\mathbf{u}}, \hat{\mathbf{u}}'\} - \mathscr{W} + \mathscr{E}_F \tag{1}$$

†Sponsored by the National Science Foundation.

‡Although the growth of delamination is a nonconservative phenomenon, the definition of an energy functional is justifiable if the growth is assumed to be monotonic at all points on the delamination edge.

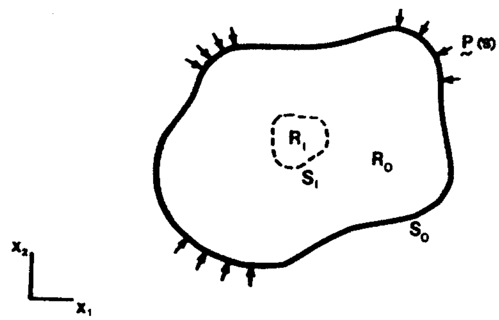

Fig. 1. Arbitrary shaped delamination embedded in edge loaded arbitrary shaped layered plate (plan view).

where

$$\mathscr{G}\{\hat{\mathbf{u}},\hat{\mathbf{u}}'\} = \int_R F(\hat{\mathbf{u}},\hat{\mathbf{u}}')\,\mathrm{d}\mathbf{x} \tag{2}$$

is the strain energy of the plate,

$$\mathscr{W} = \int_{S_0} \mathbf{P}(s)\cdot\hat{\mathbf{u}}(s)\,\mathrm{d}s \tag{3}$$

is the work done by the applied loading,

$$\mathscr{E}_F = \Gamma\left\{\int_{R_I}\mathrm{d}\mathbf{x} - \int_{R_{I_0}}\mathrm{d}\mathbf{x}\right\} \tag{4}$$

is the energy absorbed during growth of the delamination, $F(\hat{\mathbf{u}},\hat{\mathbf{u}}')$ is the strain energy density of the plate, Γ is the energy required to produce a unit area of delamination, and $(\ )'$ denotes differentiation with respect to each independent variable.

Equilibrium, under the restriction of monotonic growth of delamination, requires that the first variation of the energy functional vanish, i.e.

$$\delta\Pi = 0. \tag{5}$$

Upon examining (1)–(4) it is seen that eqn (5) requires the variation of functionals defined over a region with a moving interior boundary.

For a functional of the type

$$J\{u_1,\ldots,u_m,u_{1,x_1},\ldots,u_{m,x_n}\} = \int_R F(x_1,\ldots,x_n,u_1,\ldots,u_m,u_{1,x_1},\ldots,u_{m,x_n})\,\mathrm{d}\mathbf{x}$$

with moving boundary S, the first variation of J is given by [2]

$$\delta J = \int_R \sum_{j=1}^{m}\left[F_{,u_j} - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\frac{\partial F}{\partial u_{j,i}}\right]\overline{\delta u_j}\,\mathrm{d}\mathbf{x} + \int_R \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[\sum_{j=1}^{m}\frac{\partial F}{\partial u_{j,i}}\overline{\delta u_j} + F\delta x_i\right]\mathrm{d}\mathbf{x}, \tag{6}$$

where

$$\overline{\delta u_j} = \delta u_j - \sum_{i=1}^{n} u_{j,i}\delta x_i \tag{7}$$

(see Fig. 2) and $u_{j,i} = (\partial u_j/\partial x_i)$.

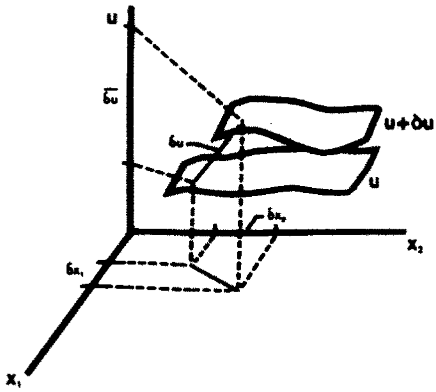

Fig. 2. Variation of function of two variables with moving boundary.

Applying the divergence theorem to the second integral in (6), and using (7), we arrive at the following expression for the variation of the functional J defined on region R with moving boundary S:

$$\delta J = \int_R \sum_{j=1}^{m} \left[F_{,u_j} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial F}{\partial u_{j,i}} \right] \overline{\delta u_j} \, \mathrm{d}\mathbf{x} + \int_S \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial F}{\partial u_{j,i}} \delta u_j \mu_i \, \mathrm{d}s$$

$$+ \int_S \sum_{i=1}^{n} \left[F \delta x_i - \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{\partial F}{\partial u_{j,i}} u_{j,k} \delta x_k \right] \mu_i \, \mathrm{d}s, \tag{8}$$

where μ is the outer normal to the surface S and s is the path coordinate along the boundary.

The results presented to this point are quite general and apply to any number of dependent variables which may correspond to deformations of a multi-layer plate, assuming an appropriate theory for each layer. For the sake of brevity we now restrict our attention to cases where it is only necessary to consider deformations of a single layer such as the case of symmetric delamination buckling as occurs in a two layer plate. For such cases it is convenient to define the array of dependent variables in the delaminated and bonded regions as follows

$$\hat{\mathbf{u}} = \begin{cases} \mathbf{u}; & x_i \in R_I \\ \mathbf{U}; & x_i \in R_0 \end{cases}, \tag{9}$$

where

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_5 \end{Bmatrix} = \begin{Bmatrix} u \\ v \\ w \\ w_{,x} \\ w_{,y} \end{Bmatrix}; \quad \mathbf{U} = \begin{Bmatrix} U_1 \\ \cdot \\ \cdot \\ \cdot \\ U_5 \end{Bmatrix} = \begin{Bmatrix} U \\ V \\ W \\ W_{,x} \\ W_{,y} \end{Bmatrix}, \tag{10}$$

and $\mathbf{x} = (x_1, x_2) = (x, y)$. In (10), u and v (U and V) correspond to the inplane layer deformations in the x and y directions respectively while w (W) corresponds to the transverse displacement of the layer centerline.†

We also define the loading vector $\mathbf{P}(s)$ as follows:

$$\mathbf{P}(s) = \begin{Bmatrix} P_1(s) \\ \cdot \\ \cdot \\ \cdot \\ P_5(s) \end{Bmatrix} = \begin{Bmatrix} P_x(s) \\ P_y(s) \\ P_z(s) \\ \mathcal{M}_x(s) \\ \mathcal{M}_y(s) \end{Bmatrix}, \tag{11}$$

†Although the development up to (8) is quite general and is valid for the approaches wherein, for example, both translational and angular displacements are used as dependent variables, we now restrict our treatment to the specific form implied by (10).

where P_x, P_y, and P_z correspond to the components of the applied distributed force in the x, y, and z directions respectively while $\mathcal{M}_x$ and $\mathcal{M}_y$ correspond to the respective components of the applied distributed bending moment.

Equation (8) in conjunction with (9)–(11) can now be used to derive the variational equations. Thus, with (8)–(11), eqn (5) becomes:

$$\int_{R_I} \sum_{j=1}^{5} \left[F_{,u_j} - \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \frac{\partial F}{\partial u_{j,i}} \right] \overline{\delta u_j}\, \mathrm{d}\mathbf{x} + \int_{S_I} \sum_{i=1}^{2} \sum_{j=1}^{5} \frac{\partial F}{\partial u_{j,i}} \delta u_j \mu_i \, \mathrm{d}s$$

$$+ \int_{R_0} \sum_{j=1}^{5} \left[F_{,U_j} - \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \frac{\partial F}{\partial U_{j,i}} \right] \overline{\delta U_j}\, \mathrm{d}\mathbf{x} - \int_{S_I} \sum_{i=1}^{2} \sum_{j=1}^{5} \frac{\partial F}{\partial U_{j,i}} \delta U_j \mu_i \, \mathrm{d}s$$

$$- \int_{S_0} \sum_{j=1}^{5} P_j \delta U_j \, \mathrm{d}s + \int_{S_0} \sum_{i=1}^{2} \sum_{j=1}^{5} \frac{\partial F}{\partial U_{j,i}} \delta U_j \mu_i \, \mathrm{d}s + \int_{S_I} \sum_{i=1}^{2} \left[F(\mathbf{u},\mathbf{u}')\delta x_i - \sum_{j=1}^{5} \sum_{k=1}^{2} \frac{\partial F}{\partial u_{j,i}} u_{j,k} \delta x_k \right] \mu_i \, \mathrm{d}s$$

$$- \int_{S_I} \sum_{i=1}^{2} \left[F(\mathbf{U},\mathbf{U}')\delta x_i - \sum_{j=1}^{5} \sum_{k=1}^{2} \frac{\partial F}{\partial U_{j,i}} U_{j,k} \delta x_k \right] \mu_i \, \mathrm{d}s + \int_{S_I} \sum_{i=1}^{2} \Gamma \delta x_i \mu_i \, \mathrm{d}s = 0. \tag{12}$$

From (10) it can be seen that only three of the five dependent variables and hence only three of the five $\overline{\delta u_j}$ (and $\overline{\delta U_j}$) can be arbitrary. We therefore seek a relationship which will allow us to express integrals of the form

$$\int_R \sum_j \mathcal{F}_j \overline{\delta u_j}\, \mathrm{d}\mathbf{x}$$

in such a way that only variations of the first three dependent variables appear in integrals over the region R. The first integral in (12) can be written as

$$\int_{R_I} \sum_{j=1}^{5} \mathcal{F}_j \overline{\delta u_j}\, \mathrm{d}\mathbf{x} = \int_{R_I} \sum_{j=1}^{2} \mathcal{F}_j \overline{\delta u_j}\, \mathrm{d}\mathbf{x} + \int_{R_I} \sum_{j=3}^{5} \mathcal{F}_j \overline{\delta u_j}\, \mathrm{d}\mathbf{x}, \tag{13}$$

where

$$\mathcal{F}_j = F_{,u_j} - \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \frac{\partial F}{\partial u_{j,i}}. \tag{14}$$

On setting $j = 3 + k$ we find the following relationship between the dependent variables

$$\left.\begin{aligned} u_3 &= w, \\ u_{3+k} &= u_{3,k} = w_{,k};\ k = 1,2. \end{aligned}\right\} \tag{15}$$

Using (15), the second integral on the r.h.s. of (13) becomes

$$\int_{R_I} \sum_{j=3}^{5} \mathcal{F}_j \overline{\delta u_j}\, \mathrm{d}\mathbf{x} = \int_{R_I} \mathcal{F}_3 \overline{\delta w}\, \mathrm{d}\mathbf{x} + \int_{R_I} \sum_{k=1}^{2} \overline{\mathcal{F}_k}\, \overline{\delta w_{,k}}\, \mathrm{d}\mathbf{x} \tag{16}$$

where

$$\overline{\mathcal{F}_k} = \mathcal{F}_{3+k}. \tag{17}$$

Further analysis is facilitated by use of the result

$$\nabla \cdot (w\mathbf{F}) = \mathbf{F} \cdot \nabla w + w \nabla \cdot \mathbf{F} \tag{18}$$

where w is a scalar and $\mathbf{F}$ is a vector. Substituting δw for w in the identity (18), rearranging terms and integrating over R we have

$$\int_R \mathbf{F}\cdot\nabla\delta w\,\mathrm{d}\mathbf{x} = \int_R \nabla\cdot(\delta w\mathbf{F})\,\mathrm{d}\mathbf{x} - \int_R \delta w\nabla\cdot\mathbf{F}\,\mathrm{d}\mathbf{x}. \tag{19}$$

On application of the divergence theorem, the above integral is transformed to

$$\int_R \mathbf{F}\cdot\nabla\delta w\,\mathrm{d}\mathbf{x} = \int_S (\mathbf{F}\cdot\boldsymbol{\mu})\delta w\,\mathrm{d}s - \int_R (\nabla\cdot\mathbf{F})\delta w\,\mathrm{d}\mathbf{x}. \tag{20}$$

The second integral on the r.h.s. of (16) can now be written as

$$\int_{R_I} \sum_{k=1}^{2} \overline{\mathcal{F}_k}\,\overline{\delta w_{,k}}\,\mathrm{d}\dot{\mathbf{x}} = \int_{S_I} \sum_{k=1}^{2} \overline{\mathcal{F}_k}\mu_k\overline{\delta w}\;\mathrm{d}s - \int_{R_I} \overline{\mathcal{F}_{k,k}}\,\overline{\delta w}\,\mathrm{d}\mathbf{x}. \tag{21}$$

Substitution of (21), (17) and (16) into (13) gives the desired form of the first integral in (12)

$$\int_{R_I} \sum_{j=1}^{5} \mathcal{F}_j\overline{\delta u_j}\,\mathrm{d}\mathbf{x} = \int_{R_I} \sum_{j=1}^{2} \mathcal{F}_j\overline{\delta u_j}\,\mathrm{d}\mathbf{x} + \int_{R_I} \left[\mathcal{F}_3 - \sum_{k=1}^{2} \mathcal{F}_{3+k,k}\right]\overline{\delta w}\,\mathrm{d}\mathbf{x} + \int_{S_I} \sum_{k=1}^{2} \mathcal{F}_{3+k}\mu_k\overline{\delta w}\,\mathrm{d}s \tag{22}$$

where $\mathcal{F}_j$ is defined by (14). A similar expression is obtained for the third integral in (12) by replacing u_j with $\underline{U_j}$.

For arbitrary $\overline{\delta u_j}(\overline{\delta U_j})$ with $j = 1,2,3$, the corresponding coefficients in the integrands of the regional integrals in (12) with (22) taken into account must vanish, resulting in the general form of the governing partial differential equations. We thus have

$$\left.\begin{aligned} &F_{,u_j} - \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\frac{\partial F}{\partial u_{j,i}} = 0; \quad j = 1,2, &(23a)\\ &F_{,w} - \sum_{k=1}^{2} \frac{\partial}{\partial x_k}\left[2\frac{\partial F}{\partial w_{,k}} - \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\frac{\partial F}{\partial w_{,ki}}\right] = 0, &(23b)\end{aligned}\right\} x_k \in R_I,$$

$$\left.\begin{aligned} &F_{,U_j} - \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\frac{\partial F}{\partial U_{j,i}} = 0; \quad j = 1,2, &(24a)\\ &U_j = 0; \quad j = 3,4,5, &(24b)\end{aligned}\right\} x_k \in R_0.$$

In a manner similar to above we arrive at the following matching and boundary conditions:

$$u_j|_{S_I} = U_j|_{S_I}; \quad j = 1,2, \tag{25a}$$

$$w|_{S_I} = w_{,k}|_{S_I} = 0; \quad k = 1,2, \tag{25b}$$

$$\left[\sum_{i=1}^{2} \frac{\partial F}{\partial u_{j,i}}\mu_i\right]_{S_I} = \left[\sum_{i=1}^{2} \frac{\partial F}{\partial U_{j,i}}\mu_i\right]_{S_I} \tag{25c}$$

$$P_j = \left[\sum_{i=1}^{2} \frac{\partial F}{\partial U_{j,i}}\mu_i\right]_{S_0}; \quad j = 1,2, \tag{25d}$$

where we take $P_j = 0$ for $j = 3$–5.

The following general form of the growth law is obtained as a consequence of the transversality condition resulting from the variations associated with the moving intermediate boundary, i.e. from the requirements that the coefficient of δx_i in (12) must vanish:

$$[\![F(\mathbf{u}, \mathbf{u}')]\!]_{S_I} - [\![\mathbf{R}\cdot\mathbf{G}]\!]_{S_I} = \Gamma. \tag{26}$$

In (26) we have used the definitions

$$[\![F(\mathbf{u},\mathbf{u}')]\!]_{S_l} \equiv [F(\mathbf{U},\mathbf{U}') - F(\mathbf{u},\mathbf{u}')]_{S_l}, \tag{27}$$

$$[\![\mathbf{R}\cdot\mathbf{G}]\!]_{S_l} \equiv \sum_{j=1}^{5}\left[\left(\sum_{k=1}^{2}\tilde{R}_{jk}\mu_k\right)\left(\sum_{i=1}^{2}\tilde{G}_{ji}\mu_i\right) - \left(\sum_{k=1}^{2}R_{jk}\mu_k\right)\left(\sum_{i=1}^{2}G_{ji}\mu_i\right)\right]_{S_l}, \tag{28}$$

$$\left.\begin{aligned} R_{jk} &\equiv \partial F/\partial u_{j,k}, \quad \tilde{R}_{jk} \equiv \partial F/\partial U_{j,k} \\ G_{jk} &\equiv u_{j,k}, \quad \tilde{G}_{jk} \equiv U_{j,k} \end{aligned}\right\} \tag{29}$$

where $[\![\ ]\!]$ indicates the "jump" in a quantity across the delamination boundary.

Equation (28) can be simplified to the following form

$$[\![\mathbf{R}\cdot\mathbf{G}]\!]_{S_l} = \left[\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{i=1}^{2}(N_{jk}D_{ji}\mu_k\mu_i - M_{jk}\Psi_{ji}\mu_k\mu_i)\right]_{S_l} \tag{30}$$

where:

$$\left.\begin{aligned} N_{jk} &= R_{jk}; \quad j = 1,2, \\ D_{jk} &= \tilde{G}_{jk} - G_{jk}; \quad j = 1,2, \end{aligned}\right\} \tag{31a}$$

$$\left.\begin{aligned} M_{jk} &= R_{ik}; \quad i = 3+j; \quad j = 1,2, \\ \Psi_{jk} &= G_{ik}; \quad i = 3+j; \quad j = 1,2. \end{aligned}\right\} \tag{31b}$$

Our analysis is now complete; its main result is the transversality condition (26)—in conjunction with this and with the assumption of monotonic growth, we have the following growth condition: if, under a given loading program, the resulting deformations are such that

$$[\![F(\mathbf{u},\mathbf{u}')]\!]_{S_l} - [\![\mathbf{R}\cdot\mathbf{G}]\!]_{S_l} > \Gamma \tag{32}$$

growth will occur with the plate finally occupying the equilibrium configuration such that (26) is satisfied at each point on the boundary of the delaminated area. No growth will occur otherwise.

The quantity D_{jk} is seen to be the (in plane) deformation gradient jump tensor and Ψ_{jk} is the (out of plane) rotation gradient tensor. It will be seen in the next section that N_{jk} is the stress resultant tensor and M_{jk} is the local bending moment tensor. The quantity $[\![\mathbf{R}\cdot\mathbf{G}]\!]_{S_l}$ therefore represents the work done by the tractions and local moments across the delamination edge due to local discontinuities. Evidently, the growth law defined by (26) and (32) is a *point wise criterion* which states that if the sum of the jump in strain energy density at, and the negative of the work done by the tractions and moments across, the delamination edge at a given point is above a certain value Γ, growth will occur with the plate evolving to a final equilibrium configuration such that a balance of the aforementioned quantities exists all along the edge.

EXPLICIT FORMS OF THE DELAMINATION GROWTH LAW

In this section the explicit form of the delamination growth law will be derived by assuming a geometrically nonlinear plate theory to govern the layer deformations.

The strain energy density for an isotropic, elastic layer is given by

$$F(\mathbf{u},\mathbf{u}') = \frac{D}{2}[w_{,xx}{}^2 + w_{,yy}{}^2 + 2\nu w_{,xx}w_{,yy} + (1-\nu)w_{,xy}{}^2]$$
$$+\frac{C}{2}\left\{\left[u_{,x}{}^2 + v_{,y}{}^2 + \frac{(1-\nu)}{2}(u_{,y}+v_{,x})^2 + 2\nu u_{,x}v_{,y}\right]\right.$$
$$+[u_{,x}w_{,x}{}^2 + v_{,y}w_{,y}{}^2 + \nu(u_{,x}w_{,y}{}^2 + v_{,y}w_{,x}{}^2) + (1-\nu)(u_{,y}+v_{,x})w_{,x}w_{,y}]$$
$$\left.+\left[\frac{1}{4}(w_{,x}{}^2 + w_{,y}{}^2)^2\right]\right\}; \quad x_i \in R_I \tag{33a}$$

$$F(\mathbf{U},\mathbf{U}') = \frac{C}{2}\left[U_{,x}{}^2 + V_{,y}{}^2 + \frac{(1-\nu)}{2}(U_{,y}+V_{,x})^2 + 2\nu U_{,x}V_{,y}\right]; \quad x_i \in R_0 \tag{33b}$$

where $D = t^2C/12$, $C = Et/(1-\nu^2)$, t is the layer thickness, E is Young's modulus for the layer and ν is Poisson's ratio for the layer. Substitution of (33) into (27) and (30) gives the following expressions for $[\![F(\mathbf{u},\mathbf{u}')]\!]_{S_I}$ and $[\![\mathbf{R}\cdot\mathbf{G}]\!]_{S_I}$:

$$[\![F(\mathbf{u},\mathbf{u}')]\!]_{S_I} = \frac{C}{2}\left\{\left[U_{,x}^2 + V_{,y}^2 + \frac{(1-\nu)}{2}(U_{,y}+V_{,x})^2 + 2\nu U_{,x}V_{,y}\right]_{S_I}\right.$$
$$\left.-\left[u_{,x}{}^2 + v_{,y}{}^2 + \frac{(1-\nu)}{2}(u_{,y}+v_{,x})^2 + 2\nu u_{,x}v_{,y}\right]_{S_I}\right\}$$
$$-\frac{D}{2}[w_{,xx}{}^2 + w_{,yy}{}^2 + 2\nu w_{,xx}w_{,yy} + (1-\nu)w_{,xy}{}^2]_{S_I} \tag{34}$$

$$[\![\mathbf{R}\cdot\mathbf{G}]\!]_{S_I} = [(N_{xx}\mu_1 + N_{xy}\mu_2)\{[\![u_{,x}]\!]\mu_1 + [\![u_{,y}]\!]\mu_2\}$$
$$+(N_{xy}\mu_1 + N_{yy}\mu_2)\{[\![v_{,x}]\!]\mu_1 + [\![v_{,y}]\!]\mu_2\}$$
$$+(M_{xy}\mu_1 + M_{yy}\mu_2)\{w_{,xy}\mu_1 + w_{,yy}\mu_2\}]_{S_I} \tag{35}$$

where

$$N_{xx}|_{S_I} = C[u_{,x} + \nu v_{,y}]_{S_I} \tag{36a}$$

$$N_{xy}|_{S_I} = C\frac{(1-\nu)}{2}[u_{,y} + v_{,x}]_{S_I} \tag{36b}$$

$$N_{yy}|_{S_I} = C[v_{,y} + \nu u_{,x}]_{S_I} \tag{36c}$$

$$M_{xx}|_{S_I} = -D[w_{,xx} + \nu w_{,yy}]_{S_I} \tag{36d}$$

$$M_{xy}|_{S_I} = -D(1-\nu)[w_{,xy}]_{S_I} \tag{36e}$$

$$M_{yy}|_{S_I} = -D[w_{,yy} + \nu w_{,xx}]_{S_I} \tag{36f}$$

Equations (26) and (32) in conjunction with (34)–(36) give the explicit form of the delamination growth law in terms of a fixed Cartesian reference frame. The explicit forms of the governing partial differential equations, matching conditions and boundary conditions can be obtained by substitution of (33) into (23)–(25).

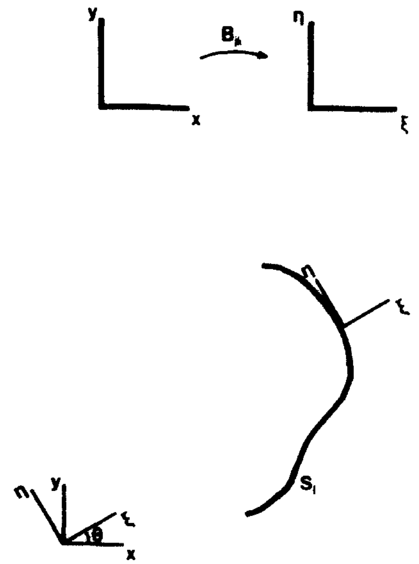

Fig. 3. Coordinate transformation and local coordinate system at delamination edge.

It is of some interest to obtain the growth law in terms of a coordinate system whose axes are aligned with the normal and tangential directions of the delamination edge. We do so in what follows.

Let B_{jk} be the orthogonal transformation which transforms the vectors in the coordinate system $\mathbf{x}$ into vectors in the coordinate system $\boldsymbol{\xi}$ at each point on S_I where $\boldsymbol{\xi}$ forms a right handed system (ξ, η) in the normal and tangential directions respectively at the delamination edge (see Fig. 3). We therefore have that

$$u_j^* = B_{jk}u_k \tag{37}$$

which relates the inplane deformations in the $\boldsymbol{\xi}$ coordinate system, u_j^*, to the inplane deformations in the original coordinate system, with

$$[B_{jk}] = [B_{jk}(s)] = \begin{bmatrix} \mu_1 & \mu_2 \\ -\mu_2 & \mu_1 \end{bmatrix}. \tag{38}$$

Similarly the resultant stress tensor, inplane deformation gradient jump tensor, local moment tensor, and out of plane rotation gradient tensor, are related to their counterparts in the $\boldsymbol{\xi}$ coordinate system by

$$\begin{aligned} N_{jk} &= B_{pj}B_{qk}N^*_{pq} \\ D_{jk} &= B_{pj}B_{qk}D^*_{pq} \\ M_{jk} &= B_{pj}B_{qk}M^*_{pq} \\ \Psi_{jk} &= B_{pj}B_{qk}\Psi^*_{pq} \end{aligned} \tag{39}$$

where a superscript * denotes the tensor components in the $\boldsymbol{\xi}$ coordinate system. Since the form of the strain energy density is invariant with respect to coordinate system, substitution of (39) into (30) allows (26) to be written as†

$$[\![F(\mathbf{u}^*, \mathbf{u}^{*\prime})]\!]_{S_I} - [\![\mathbf{R}^* \cdot \mathbf{G}^*]\!]_{S_I} = \Gamma \tag{40}$$

†Equations (40)–(42) are with respect to a cartesian reference frame whose directions are normal and tangent to the delamination edge at a *particular* point and are shown, not for computational purposes, but to bring out certain salient physical features. If one desires to investigate the relative contributions of individual modes of fracture and hence to partition the energy release into terms associated with fracture modes I, II, and III, equations (26), (32) and (35) must be written with respect to a curvilinear coordinate system whose directions run normal and parallel to the delamination edge at each point.

where

$$[\mathbf{R}^* \cdot \mathbf{G}^*]_{S_l} = [N^*_{\xi\xi}[u^*_{,\xi}] + N^*_{\xi\eta}[v^*_{,\xi}] + M^*_{\xi\xi}w_{,\xi\xi} + M^*_{\xi\eta}w_{,\xi\eta}]_{S_l} \tag{41}$$

and

$$\begin{aligned}[F(\mathbf{u}^*, \mathbf{u}^{*\prime})]_{S_l} = \frac{C}{2}\Bigg\{\Bigg[U^{*2}_{,\xi} + V^{*2}_{,\eta} + \frac{(1-\nu)}{2}(U^*_{,\eta} + V^*_{,\xi})^2 + 2\nu U^*_{,\xi}V^*_{,\eta}\Bigg]_{S_l} \\ - \Bigg[u^{*2}_{,\xi} + v^{*2}_{,\eta} + \frac{(1-\nu)}{2}(u^*_{,\eta} + v^*_{,\xi})^2 + 2\nu u^*_{,\xi}v^*_{,\eta}\Bigg]_{S_l}\Bigg\} \\ - \frac{D}{2}\Big[w^2_{,\xi\xi} + w^2_{,\eta\eta} + 2\nu w_{,\xi\xi}w_{,\eta\eta} + (1-\nu)w^2_{,\xi\eta}\Big]_{S_l}.\end{aligned} \tag{42}$$

It is seen from (41) that only the normal and shear stresses and normal bending and torsional moments at the delamination edge contribute to the growth of the delamination.

CONCLUDING REMARKS

A growth law has been derived which governs the evolution of an arbitrary shaped delamination embedded in an edge loaded, arbitrary shaped, laminated plate. In principle, this growth law can be used in conjunction with the set of governing partial differential equations, boundary and matching conditions, obtainable from the general equations and conditions given, for modeling any problem of this class where the initial delamination shape, plate shape, and loading program is prescribed. Evidently the solution of such problems will require computational approaches which are apparently yet to be developed. Chai[3] circumvented the computational difficulties for an elliptical delamination in a layer-halfspace interface by assuming that subsequent delamination shapes remain ellipses so that the functional for the debonded area can be parameterized by the lengths of the axes of the ellipses. It may also be worthwhile to note here that the growth law presented in this paper can be used in the analysis of a DCB specimen where the applied loading is nonuniform along the edge, thus, in general, resulting in a crack front which is not a straight line.

Acknowledgement—The author wishes to thank Prof. A. Maewal of Yale University for his valuable suggestions concerning the preparation of the manuscript.

REFERENCES

1. W. J. Bottega and A. Maewal, Delamination buckling and growth in laminates, *J. Appl. Mech.* (1983).
2. I. M. Gelfand and S. V. Fomin, *Calculus of Variations*, p. 176. Prentice-Hall, New Jersey (1963).
3. H. Chai, The Growth of Impact Damage in Compressively Loaded Laminates. Ph.D. Thesis, California Institute of Technology (1982).